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Notes on vector fields and transverse fields on foliated Riemannian manifolds

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0. Introduction

On a foliated Riemannian manifold with a bundle-like metric([9]), transverse Killing (conformal, affine) fields of the foliation have been studied by F.W. Kamber and Ph. Tondeur [2, 3, 4], P. Molino [5, 6], J.S. Pak and S. Yorozu [7] and others.

In this note, we investigate the properties of the transverse Killing field(t.K.f.) and the transverse conformal field(t.c.f.). Some of our results are as follows :

THEOREM A. *Let* $(M, g_M, \mathfrak{F})$ *be a Riemannian manifold with a foliation* $\mathfrak{F}$ *and a bundle-like metric* g_M *with respect to* $\mathfrak{F}$. *If* $Y \in V(\mathfrak{F})$ *is a conformal* (*resp. Killing*) *vector field on* M, *then* $s=\pi(Y)$ *is a t.c.f.* (*resp. t.K.f.*) *of* $\mathfrak{F}$.

THEOREM B. *Let* $(M, g_M, \mathfrak{F})$ *be as in Theorem A. If there exists a nonsingular t.K.f.* s *of* $\mathfrak{F}$, *then there exists a foliation* $\mathfrak{F}'$ *generated by* $\mathfrak{F}$ *and* s *such that* $\mathfrak{F} \subset \mathfrak{F}'$, codim $\mathfrak{F}'=$codim $\mathfrak{F}-1$ *and the metric* g_M *is bundle-like with respect to* $\mathfrak{F}'$.

THEOREM C. *Let* $\rho : (M, g_M) \longrightarrow (B, g_B)$ *be a Riemannian submersion with connected fibers and* $\mathfrak{F}$ *be the foliation on* M *whose leaves are fibers of the submersion* ρ. *If there exists a nonsingular t.K.f. of* $\mathfrak{F}$, *then* $\mathrm{Pont}^{(r)}(TB)=0$ *for* $r>\dim B-1$.

We shall be in C^∞-category and deal only with connected and oriented manifold without boundary. We use the following convention on the range of indices : $1 \leq i, j \leq p$, $p+1 \leq \alpha, \beta \leq p+q$. The Einstein summation convention will be used.

1. Preliminaries

Let $(M, g_M, \mathfrak{F})$ be a $p+q$ dimensional Riemannian manifold with a foliation $\mathfrak{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathfrak{F}$ ([9]). We denote by ∇ the Levi-Civita connection with respect to g_M. The foliation $\mathfrak{F}$ defines an integrable subbundle E of the tangent bundle TM over M, and let Q be the normal bundle TM/E of $\mathfrak{F}$. Let

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σ be the splitting of the exact sequence :

$$0 \longrightarrow E \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0$$

with $\sigma(Q)=E^{\perp}$, where $E^{\perp}$ denotes the orthogonal complement bundle of E in TM with respect to g_M([2]). The metric g_M induces the metric g_Q on Q, that is,

$$g_Q(s,\ t)=g_M(\sigma(s),\ \sigma(t))$$

for any $s,\ t\in\Gamma(Q)$. Here $\Gamma(Q)$ denotes the set of all sections of the bundle Q. The basic adapted (local) frame $\{X_i,\ X_\alpha\}$ of $\mathcal{F}$ is given by

$$X_i=\partial/\partial x^i,\ X_\alpha=\partial/\partial x^\alpha - A^j_\alpha \partial/\partial x^j$$

in a flat chart $U(x^i,\ x^\alpha)$ with respect to $\mathcal{F}$, where A^j_α are functions on U satisfying $g_M(X_i,\ X_\alpha)=0$([9], [10]).

Let D be the transversal Riemannian connection of $\mathcal{F}$, that is, $D:\Gamma(TM)\times\Gamma(Q)\longrightarrow\Gamma(Q)$ is given by

$$\begin{aligned} D_X s &= \pi([X,\ Y_s]) && \text{for any } X\in\Gamma(E) \\ & && \text{for any } s\in\Gamma(Q),\ \pi(Y_s)=s \\ D_X s &= \pi(\nabla_X\sigma(s)) && \text{for any } X\in\Gamma(E^{\perp}) \\ & && \text{for any } s\in\Gamma(Q) \end{aligned}$$

([2], [3]). We notice that D is torsion-free and metrical with respect to g_Q. The curvature R_D of D and the Ricci operator ρ_D of $\mathcal{F}$ are defined as follows :

$$\begin{aligned} R_D(X,\ Y)s &= D_X D_Y s - D_Y D_X s - D_{[X,\ Y]}s && \text{for any } X,\ Y\in\Gamma(TM) \\ & && \text{for any } s\in\Gamma(Q) \\ \rho_D(s) &= g^{\alpha\beta}R_D(\sigma(s),\ \pi(X_\alpha))\pi(X_\beta) && \text{for any } s\in\Gamma(Q) \end{aligned}$$

where $(g^{\alpha\beta})$ denotes the inverse matrix of $(g_{\alpha\beta})$ with $g_{\alpha\beta}=g_M(X_\alpha,\ X_\beta)$ ([2],[3],[7]). Let $V(\mathcal{F})$ be the set of all infinitesimal automorphisms of $\mathcal{F}$, that is, $V(\mathcal{F})=\{Y\in\Gamma(TM) \mid [Y,\ Z]\in\Gamma(E)$ for any $Z\in\Gamma(E)\}$, and we set $\overline{V}(\mathcal{F})=\{s\in\Gamma(Q) \mid s=\pi(Y)$ for any $Y\in V(\mathcal{F})\}$ ([2]). Then we have some operators ([2],[3],[5],[6],[7]) :

(i) the transverse Lie differentiation $\Theta(Y)$ with respect to $Y\in V(\mathcal{F})$, $\Theta(Y)s=\pi([Y,\ Y_s])$, where $\pi(Y_s)=s$,

(ii) the transverse divergence div_D, $\mathrm{div}_D s=g^{\alpha\beta}g_Q(D_{X_\alpha}s,\ \pi(X_\beta))$,

(iii) the transverse gradient grad_D, $\mathrm{grad}_D f=g^{\alpha\beta}X_\alpha(f)\pi(X_\beta)$,

(iv) the Laplacian $\Delta_D=d_D\,d_D{}^*+d_D{}^*\,d_D$.

The second fundamental form α of $\mathcal{F}$ is given by

$$\alpha(X,\ Y)=-(D_X\,\pi)(Y) \quad \text{for any } X,\ Y\in\Gamma(TM),$$

and the tension field τ of $\mathcal{F}$ is given by

$$\tau=g^{ji}\alpha(X_i,\ X_j)(=d_D{}^*\pi),$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}) with $g_{ij}=g_M(X_i, X_j)$ ([2], [3]). The foliation $\mathcal{F}$ is called harmonic if $\tau=0$, and $\mathcal{F}$ is called totally geodesic if $\alpha=0$([2], [3]).

2. Transverse fields

Let $(M, g_M, \mathcal{F})$ be as in section 1. The definitions of Killing vector field, conformal vector field, and affine vector field on M are abbreviated. In order to give the definitions of geometric transverse fields of $\mathcal{F}$, we have to show the following lemma :

LEMMA 2.1. *For any $X\in\Gamma(E)$, it holds that $\Theta(X)g_Q=0$ and $\Theta(X)D=0$.*

This lemma is proved by the direct calculation.

DEFINITION 2.2([2], [3], [5], [6], [7]). If $Y\in V(\mathcal{F})$ satisfies $\Theta(Y)g_Q=0$, then $s=\pi(Y)$ is called a transverse Killing field (t.K.f.) of $\mathcal{F}$. If $Y\in V(\mathcal{F})$ satisfies $\Theta(Y)g_Q=2f\cdot g_Q$ for some function f on M, then $s=\pi(Y)$ is called a transverse conformal field (t.c.f.) of $\mathcal{F}$, and $f_s=f$ is called the characteristic function of s. If $Y\in V(\mathcal{F})$ satisfies $\Theta(Y)D=0$, then $s=\pi(Y)$ is called a transverse affine field (t.a.f.) of $\mathcal{F}$.

Then we have

THEOREM 2.3([7]).

(*a*) *If s is a t.a.f. of $\mathcal{F}$, then $\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s)$ and $\mathrm{div}_D s=const.$*

(*b*) *If s is a t.K.f. of $\mathcal{F}$, then $\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s)$ and $\mathrm{div}_D s=0$.*

(*b*) *If s is a t.c.f. of $\mathcal{F}$, then $\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s)+(1-\frac{2}{q})\,\mathrm{grad}_D\mathrm{div}_D s$ and $f_s=\frac{1}{q}\mathrm{div}_D s$.*

REMARK 2.4([7]). For any $s\in \overline{V}(\mathcal{F})$, $\mathrm{div}_D s$ is a foliated function on M.

We set

$K(\mathcal{F})$: the set of all transverse Killing fields of $\mathcal{F}$

$C(\mathcal{F})$: the set of all transverse conformal fields of $\mathcal{F}$

$A(\mathcal{F})$: the set of all transverse affine fields of $\mathcal{F}$.

For any s, $t\in \overline{V}(\mathcal{F})$, we define $[s, t]$ by

$$[s, t]=\pi([Y_s, Y_t]) \qquad \pi(Y_s)=s \text{ with } Y_s\in V(\mathcal{F})$$
$$\pi(Y_t)=t \text{ with } Y_t\in V(\mathcal{F}).$$

Then we have

THEOREM 2.5. *$K(\mathcal{F})$, $C(\mathcal{F})$ and $A(\mathcal{F})$ are Lie algebras with respect to the bracket defined as above.*

Now, we have

LEMMA 2.6. *For any $Y\in V(\mathcal{F})$, u, $v\in\Gamma(Q)$, it holds that $(\Theta(Y)g_Q)(u, v)=(\mathcal{L}(Y)g_M)(\sigma(u),\sigma(v))$, where $\mathcal{L}(Y)$ denotes the Lie differentiation with respect to Y.*

If f_Y is a characteristic function of a conformal vector field $Y \in V(\mathcal{F})$, then, by Lemma 2.6, f_Y is a foliated function on M. Thus we have

THEOREM 2.7. *If $Y \in V(\mathcal{F})$ is a conformal vector field on M, then $s=\pi(Y)$ is a t.c.f. of $\mathcal{F}$ whose characteristic function is one of Y.*

COROLLARY 2.8. *If $Y \in V(\mathcal{F})$ is a Killing vector field on M, then $s=\pi(Y)$ is a t.K.f. of $\mathcal{F}$.*

We denote by Y_E (resp. $Y_{E^\perp}$) the E (resp. $E^\perp$)$-$component of a vector field Y on M. The following results are easily proved

PROPOSITION 2.9. *Let $Y \in V(\mathcal{F})$ be a conformal vector field on M. Then Y_E is a Killing vector field on M if and only if $Y_{E^\perp}$ is a conformal vector field on M whose characteristic function is one of Y.*

PROPOSITION 2.10. *Let $X \in \Gamma(E)$ be a Killing vector field on M and $Z \in V(\mathcal{F}) \cap \Gamma(E^\perp)$ a conformal vector field on M. Then $Y=X+Z$ is a conformal vector field on M whose characteristic function is one of Z.*

Next, by Jacobi identity, we have that $(\Theta(Y)D)_X t=0$ for any $Y \in V(\mathcal{F})$, $X \in \Gamma(E)$, and $t \in \Gamma(Q)$. Moreover, we have

PROPOSITION 2.11. *For any $Y \in V(\mathcal{F}) \cap \Gamma(E^\perp)$, $V \in \Gamma(E^\perp)$, and $t \in \Gamma(Q)$, it holds that $(\Theta(Y)D)_V t=\pi((\mathcal{L}_Y\nabla)_V\sigma(t))+\pi(\nabla_{\sigma(t)}(\mathcal{L}_Y V)_E+\nabla_V(\mathcal{L}_Y\sigma(t))_E)$.*

The local expression of $\pi(\nabla_{\sigma(t)}(\mathcal{L}_Y V)_E)$ is given by $-\frac{1}{2}Y^\alpha V^\beta Z^\tau B^k_{\alpha\beta}B^h_{\varepsilon\tau}\, g^{\gamma\varepsilon}g_{kh}\,\pi(X_\gamma)$, where $Y=Y^\alpha X_\alpha$, $V=V^\beta X_\beta$, $\sigma(t)=Z^\tau X_\tau$, and $[X_\alpha, X_\beta]=B^k_{\alpha\beta}X_k$. Thus we have

PROPOSITION 2.12. *Suppose that $E^\perp$ is an integrable subbundle of TM. If $Y \in V(\mathcal{F}) \cap \Gamma(E^\perp)$ is an affine vector field on M, then $s=\pi(Y)$ is a t.a.f. of $\mathcal{F}$.*

REMARK 2.13. For a t.K.f. (resp. t.c.f., t.a.f.) s of $\mathcal{F}$, $\sigma(s)$ (or Y with $\pi(Y)=s$) is not always a Killing (resp. conformal, affine) vector field on M.

3. Nonsingular transverse Killing fields

Let $(M, g_M, \mathcal{F})$ be as in section 2. We consider a nonsingular t.K.f. s of $\mathcal{F}$ with $s=\pi(Y) \in \overline{V}(\mathcal{F})$. Let E' be a subbundle of TM generated by E and Y. Then E' is an integrable subbundle. Thus we have a new foliation $\mathcal{F}'$ defined by E', and codim $\mathcal{F}'=q-1$. For any $u, v \in \Gamma(Q)$, the equality $(\Theta(Y)g_Q)(u, v)=0$ implies $g_M(\nabla_{\sigma(u)}Y, \sigma(v))+g_M(\sigma(u), \nabla_{\sigma(v)}Y)=0$. And we easily have that $g_M(\nabla_{\sigma(u)}Y_E, \sigma(v))+g_M(\sigma(u), \nabla_{\sigma(v)}Y_E)=0$, where Y_E denotes the E-component of Y. Thus we have that $g_M(\nabla_{\sigma(u)}\sigma(s), \sigma(v))+g_M(\sigma(u), \nabla_{\sigma(v)}\sigma(s))=0$. Here $\sigma(s)$ is the $E^\perp$-component of Y. Therefore, by Theorem 3.1

in[10], we have

THEOREM 3.1([1]). *Let $(M, g_M, \mathcal{F})$ be as above. If there exists a nonsingular t.K.f. of $\mathcal{F}$, then there exists a foliation $\mathcal{F}'$ such that $\mathcal{F} \subset \mathcal{F}'$ and* $\mathrm{codim}\,\mathcal{F}' = \mathrm{codim}\,\mathcal{F} - 1$, *and the metric g_M is bundle-like with respect to $\mathcal{F}'$*

Let $\rho : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion with connected fibers. Then we have a foliation $\mathcal{F}$ on M whose leaves are fibers $\rho^{-1}(b)$ with $b \in B$. If $s = \pi(Y) \in \overline{V}(\mathcal{F})$ is a nonsingular t.K.f. of $\mathcal{F}$, then $\rho_*(Y)$ is a nonsingular Killing vector field on B and defines an integrable subbundle $\{\rho_*(Y)\}$ of the tangent bundle TB over B. Then, by Pasternack's theorem[8], we have

THEOREM 3.2([1]). *Let $\rho : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian submersion with connected fibers and $\mathcal{F}$ be the foliation on M whose leaves are fibers of the submersion ρ. If there exists a nonsingular t.K.f. of $\mathcal{F}$, then* $\mathrm{Pont}^{(r)}(TB) = 0$ *for* $r > \dim B - 1$, *where* $\mathrm{Pont}^{(*)}(TB)$ *denotes the characteristic ring generated by the real Pontryagin classes of TB.*

PROOF. Let $s = \pi(Y) \in \overline{V}(\mathcal{F})$ be a nonsingular t.K.f. of $\mathcal{F}$. Then $\rho_*(Y)$ defines an integrable subbundle $\{\rho_*(Y)\}$ of TB and a foliation $\mathcal{F}_B$ on B. The codimension of $\mathcal{F}_B$ is equal to $\dim B - 1$. Since $\rho_*(Y)$ is a Killing vector field on B, g_B is a bundle-like metric with respect to $\mathcal{F}_B$. Thus, by Pasternack's vanishing theorem, we have that $\mathrm{Pont}^{(r)}(TB) = 0$ for $r > \dim B - 1$.

4. Harmonic foliations and transverse Killing fields

Let $(M, g_M, \mathcal{F})$ be as in section 1. We have the following theorems :

THEOREM 4.1([7]). *Suppose that M is compact and $\mathcal{F}$ is harmonic. Then it holds that* $\int_M \mathrm{div}_D\, s\, dM = 0$ *for any* $s \in \Gamma(Q)$.

THEOREM 4.2([7]). *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1. If $s \in \overline{V}(\mathcal{F})$ satisfies* $\Delta_D s = \rho_D(s)$ *and* $\mathrm{div}_D s = 0$, *then s is a t.K.f. of $\mathcal{F}$.*

Then we have

THEOREM 4.3([7]). *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1. Then every t.a.f. of $\mathcal{F}$ is a t.K.f. of $\mathcal{F}$.*

THEOREM 4.4 *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1. Suppose that $\mathcal{F}$ has dense leaves. Then every t.c.f. of $\mathcal{F}$ is a t.K.f. of $\mathcal{F}$.*

PROOF. Let s be a t.c.f. of $\mathcal{F}$. By Theorem 2.3 and Remark 2.4, $\mathrm{div}_D\, s = q \cdot f_s$ is a foliated function on M. Since $\mathcal{F}$ has dense leaves, $\mathrm{div}_D\, s$ is a constant function on M. Thus, by Theorem 4.1, $\mathrm{div}_D\, s = 0$. Therefore, Theorem 4.2 implies that s is a t.K.f. of $\mathcal{F}$.

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